(8)

Exercise 1.5.15

$$x^{\frac{1}{n}} y^{\frac{1}{n}} = (xy)^{\frac{1}{n}}$$

Lemma

$(x^a)^b = x$ if $a = b^{-1}$

$(x^a)^b = x^{ab}$ (computational)

$x^{ab} = x$ if $ab = 1$

$ab = 1$ if $a = b^{-1}$ (uniqueness of b^{-1})

$$\therefore ((xy)^{\frac{1}{n}})^n = xy$$

Lemma

$x^a \cdot x^b = x^{a+b}$ ~~(Lemma)~~ (computational)

$$(x^{\frac{1}{n}} y^{\frac{1}{n}})^n$$

$$= x^{\frac{1}{n}+\frac{1}{n}+\dots+\frac{1}{n}} \cdot y^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\dots+\frac{1}{n}}$$

$$= x^{\frac{n}{n}} \cdot y^{\frac{n}{n}}$$

$$= xy$$

$$\therefore ((xy)^{\frac{1}{n}})^n = (x^{\frac{1}{n}} y^{\frac{1}{n}})^n$$

Now, we showed in 1.5.13 that for a complete ordered field, with $n \in \mathbb{N}$, $x > 0$ in F. Then there exists a unique $y \in F$ such that $y > 0$ and $y^n = x$. We denote this y by $y = \sqrt[n]{x}$.

So then $(xy)^{\frac{1}{n}}$ is the only positive element such that $((xy)^{\frac{1}{n}})^n = xy$, and so $(xy)^{\frac{1}{n}} = x^{\frac{1}{n}} y^{\frac{1}{n}}$